

INVERSE STURM-LIOUVILLE PROBLEMS WITH FIXED BOUNDARY CONDITIONS

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ABSTRACT. Necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectral data for a certain Sturm-Liouville problem are well known. We add two more conditions so that the same two sequences become necessary and sufficient for being the spectral data for a Sturm-Liouville problem with fixed boundary conditions.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let us denote by $L(q, \alpha, \beta)$ the Sturm-Liouville boundary-value problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0, \pi), \quad \mu \in \mathbb{C}, \quad (1.1)$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad \alpha \in (0, \pi], \quad (1.2)$$

$$y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \quad \beta \in [0, \pi), \quad (1.3)$$

where q is a real-valued functions which are integrable on $[0, \pi]$ (we write $q \in L^1_{\mathbb{R}}[0, \pi]$). By $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by problem (1.1)-(1.3) (see [11]). It is known, that under these conditions the spectra of the operator $L(q, \alpha, \beta)$ is discrete and consists of real, simple eigenvalues [11], which we denote by $\mu_n = \mu_n(q, \alpha, \beta) = \lambda_n^2(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, emphasizing the dependence of μ_n on q , α and β . We assume that eigenvalues are enumerated in the increasing order, i.e.,

$$\mu_0(q, \alpha, \beta) < \mu_1(q, \alpha, \beta) < \dots < \mu_n(q, \alpha, \beta) < \dots$$

Let $\varphi(x, \mu, \alpha, q)$ and $\psi(x, \mu, \beta, q)$ be the solutions of the equation (1.1), which satisfy the initial conditions

$$\varphi(0, \mu, \alpha, q) = \sin \alpha, \quad \varphi'(0, \mu, \alpha, q) = -\cos \alpha,$$

$$\psi(\pi, \mu, \beta, q) = \sin \beta, \quad \psi'(\pi, \mu, \beta, q) = -\cos \beta,$$

respectively. The eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, of $L(q, \alpha, \beta)$ are the solutions of the equation

$$\Phi(\mu) = \Phi(\mu, \alpha, \beta) := \varphi(\pi, \mu, \alpha) \cos \beta + \varphi'(\pi, \mu, \alpha) \sin \beta = 0,$$

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or of the equation

$$\Psi(\mu) = \Psi(\mu, \alpha, \beta) := \psi(0, \mu, \beta) \cos \alpha + \psi'(0, \mu, \beta) \sin \alpha = 0.$$

According to the well-known Liouville formula, the wronskian $W(x) = W(x, \varphi, \psi) = \varphi\psi' - \varphi'\psi$ of the solutions φ and ψ is constant. It follows that $W(0) = W(\pi)$ and, consequently $\Psi(\mu, \alpha, \beta) = -\Phi(\mu, \alpha, \beta)$. It is easy to see that the functions $\varphi_n(x) := \varphi(x, \mu_n, \alpha, q)$ and $\psi_n(x) := \psi(x, \mu_n, \beta, q)$, $n = 0, 1, 2, \dots$, are the eigenfunctions, corresponding to the eigenvalue μ_n . Since all eigenvalues are simple, there exist constants $c_n = c_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, such that

$$\varphi_n(x) = c_n \psi_n(x). \quad (1.4)$$

The squares of the L^2 -norm of these eigenfunctions:

$$a_n = a_n(q, \alpha, \beta) := \int_0^\pi |\varphi_n(x)|^2 dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = b_n(q, \alpha, \beta) := \int_0^\pi |\psi_n(x)|^2 dx, \quad n = 0, 1, 2, \dots$$

are called norming constants.

In this article we consider the case $\alpha, \beta \in (0, \pi)$; i.e. we assume that $\sin \alpha \neq 0$ and $\sin \beta \neq 0$. In this case we consider the solution $\tilde{\varphi}(x, \mu, \alpha, q) := \frac{\varphi(x, \mu, \alpha, q)}{\sin \alpha}$ of (1.1) which has the initial values

$$\tilde{\varphi}(0, \mu, \alpha, q) = 1, \quad \tilde{\varphi}'(0, \mu, \alpha, q) = -\cot \alpha;$$

also we consider the solution $\tilde{\psi}(x, \mu, \beta, q) := \frac{\psi(x, \mu, \beta, q)}{\sin \beta}$. Of course, the functions $\tilde{\varphi}_n(x) := \tilde{\varphi}(x, \mu_n, \alpha, q)$ and $\tilde{\psi}_n(x) := \tilde{\psi}(x, \mu_n, \beta, q)$, $n = 0, 1, 2, \dots$, are the eigenfunctions, corresponding to the eigenvalue μ_n . It follows from (1.4) that for norming constants $\tilde{a}_n := \|\tilde{\varphi}_n\|^2 = \frac{a_n}{\sin^2 \alpha}$ and $\tilde{b}_n := \|\tilde{\psi}_n\|^2 = \frac{b_n}{\sin^2 \beta}$ satisfy

$$\tilde{b}_n = \frac{b_n}{\sin^2 \beta} = \frac{a_n}{c_n^2 \sin^2 \beta} = \frac{\tilde{a}_n \sin^2 \alpha}{c_n^2 \sin^2 \beta}. \quad (1.5)$$

The inverse problem by “spectral function” (see [1, 2, 3, 7, 9, 10, 12, 13]) is the reconstruction of the problem (q, α, β) from the spectra $\{\mu_n\}_{n=0}^\infty$ and the norming constants $\{\tilde{a}_n\}_{n=0}^\infty$ (or $\{\tilde{b}_n\}_{n=0}^\infty$). The two sequences $\{\mu_n\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ together will be called the spectral data.

In this article we state the question

What kind of sequences $\{\mu_n\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ can be the spectral data for problem (q, α, β) with $q \in L^2_{\mathbb{R}}[0, \pi]$ and in advance fixed α and β in $(0, \pi)$?

Our answer is in the following assertion.

Theorem 1.1. *For a real increasing sequence $\{\mu_n\}_{n=0}^\infty$ and a positive sequence $\{\tilde{a}_n\}_{n=0}^\infty$ to be spectral data for boundary-value problem (q, α, β) with a $q \in L^2_{\mathbb{R}}[0, \pi]$ and fixed $\alpha, \beta \in (0, \pi)$ it is necessary and sufficient that the following relations hold:*

$$\lambda_n = \sqrt{\mu_n} = n + \frac{\omega}{\pi n} + \frac{\omega_n}{n}, \quad \omega = \text{const}, \quad \{\omega_n\}_{n=0}^\infty \in l^2, \quad (1.6)$$

$$\tilde{a}_n = \frac{\pi}{2} + \frac{\kappa_n}{n}, \quad \{\kappa_n\}_{n=0}^\infty \in l^2, \quad (1.7)$$

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^\infty \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi} \right) = \cot \alpha, \quad (1.8)$$

$$\frac{\tilde{a}_0}{\pi^2 \cdot \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}\right)^2} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{\tilde{a}_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k \neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{2}{\pi} \right) = -\cot \beta. \quad (1.9)$$

To prove Theorem 1.1 we prove the following assertion, which has independent interest.

Theorem 1.2. *Let $q \in L^2_{\mathbb{R}}[0, \pi]$ and $\alpha, \beta \in (0, \pi)$. Then for norming constants $\tilde{a}_n = \tilde{a}_n(q, \alpha, \beta)$ and $\tilde{b}_n = \tilde{b}_n(q, \alpha, \beta)$ satisfy*

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi} \right) = \cot \alpha, \quad (1.10)$$

$$\frac{1}{\tilde{b}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi} \right) = -\cot \beta. \quad (1.11)$$

Let us note that asymptotic behavior of $\{\mu_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ are standard conditions for the solution of the inverse problem. The conditions (1.8) and (1.9) which we add to the conditions (1.6) and (1.7) guarantee that α and β , which we construct during the solution of the inverse problem, are the same that we fixed in advance. At the same time Theorem 1.2 says that the conditions (1.8) and (1.9) are necessary.

2. PROOF OF THEOREM 1.2

The solution $\tilde{\varphi}$ has the well known representation (see [1, 2, 3, 9, 10])

$$\tilde{\varphi}(x, \lambda, \alpha, q) = \cos \lambda x + \int_0^x G(x, t) \cos \lambda t dt, \quad (2.1)$$

where about the kernel $G(x, t)$ we know (in particular) that

$$G(x, x) = -\cot \alpha + \frac{1}{2} \int_0^x q(s) ds. \quad (2.2)$$

It is also known that $G(x, t)$ satisfies to the Gelfand-Levitan integral equation

$$G(x, t) + F(x, t) + \int_0^x G(x, s) F(s, t) ds = 0, \quad 0 \leq t \leq x, \quad (2.3)$$

where (see [1])

$$F(x, t) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x \cos \lambda_n t}{\tilde{a}_n} - \frac{\cos nx \cos nt}{a_n^0} \right) \quad (2.4)$$

where $a_0^0 = \pi$ and $a_n^0 = \frac{\pi}{2}$ for $n = 1, 2, \dots$. From (2.2)–(2.4) it follows that

$$\begin{aligned} G(0, 0) &= -F(0, 0) = -\sum_{n=0}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{1}{a_n^0} \right) \\ &= -\left(\frac{1}{\tilde{a}_0} - \frac{1}{\pi} \right) - \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi} \right) = -\cot \alpha. \end{aligned} \quad (2.5)$$

Thus, (1.10) is proved.

Let us now consider the functions (compare with [8])

$$p(x, \mu_n) = \frac{\varphi(\pi - x, \mu_n, \alpha, q)}{\varphi(\pi, \mu_n, \alpha, q)} = \frac{\varphi(\pi - x, \mu_n)}{\varphi(\pi, \mu_n)}, \quad n = 0, 1, 2, \dots \quad (2.6)$$

Since $\varphi(x, \mu, \alpha, q)$ satisfies (1.1), and

$$p'(x, \mu_n) = -\frac{\varphi'(\pi - x, \mu_n)}{\varphi(\pi, \mu_n)}, \quad p''(x, \mu_n) = \frac{\varphi''(\pi - x, \mu_n)}{\varphi(\pi, \mu_n)},$$

we can see that $p(x, \mu_n)$ satisfies

$$-p''(x, \mu_n) + q(\pi - x)p(x, \mu_n) = \mu_n p(x, \mu_n)$$

and the initial conditions

$$p(0, \mu_n) = 1, \quad p'(0, \mu_n) = -\frac{\varphi'(\pi, \mu_n)}{\varphi(\pi, \mu_n)} = -(-\cot \beta) = \cot \beta = -\cot(\pi - \beta). \quad (2.7)$$

Also we have

$$\begin{aligned} p(\pi, \mu_n) &= \frac{\varphi(0, \mu_n)}{\varphi(\pi, \mu_n)} = \frac{\sin \alpha}{\varphi(\pi, \mu_n)} = \frac{\sin(\pi - \alpha)}{\varphi(\pi, \mu_n)}, \\ p'(\pi, \mu_n) &= -\frac{\varphi'(0, \mu_n)}{\varphi(\pi, \mu_n)} = -\frac{-\cos \alpha}{\varphi(\pi, \mu_n)} = \frac{-\cos(\pi - \alpha)}{\varphi(\pi, \mu_n)}. \end{aligned}$$

It follows, that $p_n(x) := p(x, \mu_n)$ satisfy to the boundary condition

$$p_n(\pi) \cos(\pi - \alpha) + p'_n(\pi) \sin(\pi - \alpha) = 0, \quad n = 0, 1, 2, \dots$$

Let us denote $q^*(x) := q(\pi - x)$. Since $\mu_n(q^*, \pi - \beta, \pi - \alpha) = \mu_n(q, \alpha, \beta)$ (it is easy to prove and is well known [7]), it follows, that $p_n(x)$, $n = 0, 1, 2, \dots$, are the eigenfunctions of problem $(q^*, \pi - \beta, \pi - \alpha)$, which have the initial conditions (2.7); i.e. $p_n(x) = \tilde{\varphi}(x, \mu_n, \pi - \beta, q^*)$, $n = 0, 1, 2, \dots$

Thus, as in (2.5), for norming constants $\hat{a}_n = \|p(\cdot, \mu_n)\|^2$ must satisfy

$$\left(\frac{1}{\hat{a}_0} - \frac{1}{\pi}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{\hat{a}_n} - \frac{2}{\pi}\right) = \cot(\pi - \beta) = -\cot \beta. \quad (2.8)$$

On the other hand, for the norming constants \hat{a}_n , according to (1.4), (1.5) and (2.6), we have

$$\begin{aligned} \hat{a}_n &= \int_0^\pi p^2(x, \mu_n) dx \\ &= \int_0^\pi \frac{\varphi^2(\pi - x, \mu_n)}{\varphi^2(\pi, \mu_n)} dx \\ &= -\frac{1}{\varphi^2(\pi, \mu_n)} \int_\pi^0 \varphi^2(s, \mu_n) ds \\ &= \frac{1}{\varphi^2(\pi, \mu_n)} \int_0^\pi \varphi^2(s, \mu_n) ds \\ &= \frac{a_n(q, \alpha, \beta)}{\varphi^2(\pi, \mu_n)} = \frac{\tilde{a}_n \sin^2 \alpha}{c_n^2 \sin^2 \beta} = \tilde{b}_n. \end{aligned}$$

Therefore, we can rewrite (2.8) in the form

$$\left(\frac{1}{\tilde{b}_0} - \frac{1}{\pi}\right) - \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = \cot(\pi - \beta) = -\cot \beta.$$

Thus, (1.11) holds, and Theorem 1.2 is proved.

Let us note that the specification of the spectra $\{\mu_n(q, \alpha, \beta)\}_{n=0}^\infty$ (of a problem (q, α, β)) uniquely determines the characteristic function $\Phi(\mu)$ (see [4, Lemma 2.2],

see also [7, Lemma 1]), and its derivative $\frac{\partial \Phi(\mu)}{\partial \mu} = \dot{\Phi}(\mu)$ (see [4, lemma2.3]). In particular, if $\alpha, \beta \in (0, \pi)$ the following formulae hold:

$$\dot{\Phi}(\mu_0) = -\pi \sin \alpha \sin \beta \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}, \quad (2.9)$$

$$\dot{\Phi}(\mu_n) = -\frac{\pi}{n^2} [\mu_0 - \mu_n] \sin \alpha \sin \beta \prod_{k=1, k \neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}, \quad (2.10)$$

for $n = 1, 2, \dots$. On the other hand, it is easy to prove the relation (see [4, (2.16) in Lemma 2.2] and [7, Lemma 1])

$$a_n = -c_n \dot{\Phi}(\mu_n). \quad (2.11)$$

To take into account the relations (1.5) and (2.9)-(2.11) we find formulae for $1/\tilde{b}_0$ and $1/\tilde{b}_n$ with $n = 1, 2, \dots$ (in terms of $\{\mu_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$):

$$\frac{1}{\tilde{b}_0} = \frac{\tilde{a}_0}{\pi^2 \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2} \right)^2}, \quad (2.12)$$

$$\frac{1}{\tilde{b}_n} = \frac{\tilde{a}_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k \neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2}. \quad (2.13)$$

So, we can change the second assertion in Theorem 1.2 by the assertion

$$\begin{aligned} & \frac{\tilde{a}_0}{\pi^2 \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2} - \frac{1}{\pi} \\ & + \sum_{n=1}^{\infty} \left(\frac{\tilde{a}_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k \neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2} - \frac{2}{\pi} \right) = -\cot \beta, \end{aligned}$$

which coincides with (1.9).

3. PROOF OF THE THEOREM 1.1

For μ_n we have proved in [5] (in a more general case, when $q \in L^1_{\mathbb{R}}[0, \pi]$) the asymptotic formula

$$\mu_n(q, \alpha, \beta) = [n + \delta_n(\alpha, \beta)]^2 + \frac{1}{\pi} \int_0^{\pi} q(t) dt + r_n(q, \alpha, \beta), \quad (3.1)$$

where δ_n is the solution of the equation

$$\begin{aligned} \delta_n(\alpha, \beta) &= \frac{1}{\pi} \arccos \frac{\cos \alpha}{\sqrt{[n + \delta_n(\alpha, \beta)]^2 \sin^2 \alpha + \cos^2 \alpha}} \\ &\quad - \frac{1}{\pi} \arccos \frac{\cos \beta}{\sqrt{[n + \delta_n(\alpha, \beta)]^2 \sin^2 \beta + \cos^2 \beta}} \end{aligned} \quad (3.2)$$

and $r_n(q, \alpha, \beta) = o(1)$, when $n \rightarrow \infty$, uniformly in $\alpha, \beta \in [0, \pi]$ and q from any bounded subset of $L^1_{\mathbb{R}}[0, \pi]$ (we will write $q \in BL^1_{\mathbb{R}}[0, \pi]$). It follows from (3.2) (see [5] for details), that if $\sin \alpha \neq 0$ and $\sin \beta \neq 0$, ($\alpha, \beta \in (0, \pi)$), then

$$\delta_n(\alpha, \beta) = \frac{\cot \beta - \cot \alpha}{\pi n} + O\left(\frac{1}{n^2}\right). \quad (3.3)$$

It is not difficult to obtain from (3.1) that (see [6])

$$\lambda_n = \sqrt{\mu_n} = n + \delta_n(\alpha, \beta) + \frac{[q]}{2[n + \delta_n(\alpha, \beta)]} + l_n + O\left(\frac{1}{n^2}\right), \quad (3.4)$$

where

$$l_n = \frac{1}{\pi[n + \delta_n(\alpha, \beta)]} \int_0^\pi q(x) \cos 2\lambda_n x dx = o\left(\frac{1}{n}\right)$$

and $[q] = \frac{1}{\pi} \int_0^\pi q(t) dt$.

In the case $q \in L^2_{\mathbb{R}}[0, \pi]$ and $\alpha, \beta \in (0, \pi)$ it follows from (3.3) and (3.4) that $l_n = \omega_n/n$, where $[\omega_n] \in l^2$ and we can rewrite (3.4) in the form

$$\lambda_n = n + \frac{\omega}{n} + \frac{\omega_n}{n}, \quad (3.5)$$

where $\omega = \text{const} = (\cot \beta - \cot \alpha + \frac{\pi}{2}[q])/\pi$ and $\{\omega_n\}_{n=0}^\infty \in l^2$, i.e. $\sum_{n=1}^\infty |\omega_n|^2 < \infty$. In [1] there is a proof of such assertion:

Theorem 3.1 ([1]). *For real numbers $\{\lambda_n^2\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ to be the spectral data for a certain boundary-value problem (q, α, β) with $q \in L^2_{\mathbb{R}}[0, \pi]$, $(\alpha, \beta \in (0, \pi))$, it is necessary and sufficient that relations (1.6) and (1.7) hold.*

Thus, if we have a real sequence $\{\mu_n\}_{n=0}^\infty = \{\lambda_n^2\}_{n=0}^\infty$, which has the asymptotic representation (1.6) and a positive sequence $\{\tilde{a}_n\}_{n=0}^\infty$, which has the asymptotic representation (1.7), then, according to the Theorem 3.1, there exist a function $q \in L^2_{\mathbb{R}}[0, \pi]$ and some constants $\tilde{\alpha}, \tilde{\beta} \in (0, \pi)$ such that λ_n^2 , $n = 0, 1, 2, \dots$, are the eigenvalues and \tilde{a}_n , $n = 0, 1, 2, \dots$, are norming constants of a Sturm-Liouville problem $(q, \tilde{\alpha}, \tilde{\beta})$.

The function $q(x)$ and constants $\tilde{\alpha}, \tilde{\beta}$ are obtained on the way of solving the inverse problem by Gel'fand-Levitan method. The algorithm of that method is as follows:

First we define the function $F(x, t)$ by formula (2.4) (note that this function is defined by $\{\lambda_n\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ uniquely). Then we consider the integral equation (2.3), where $G(x, \cdot)$ is unknown function. It is proved (see [1]) that provided (3.5) and (1.7) the integral equation (2.3) has a unique solution $G(x, t)$. With function $G(x, t)$, we construct a function

$$\tilde{\varphi}(x, \lambda) = \cos \lambda x + \int_0^x G(x, t) \cos \lambda t dt, \quad (3.6)$$

which is defined for all $\lambda \in \mathbb{C}$. It is proved (see [1]) that

$$-\tilde{\varphi}''(x, \lambda^2) + \left(2 \frac{d}{dx} G(x, x)\right) \tilde{\varphi}(x, \lambda^2) = \lambda^2 \tilde{\varphi}(x, \lambda^2), \quad (3.7)$$

almost everywhere on $(0, \pi)$,

$$\begin{aligned} \tilde{\varphi}(0, \lambda^2) &= 1, \\ \tilde{\varphi}'(0, \lambda^2) &= G(0, 0). \end{aligned}$$

If we state the condition

$$G(0, 0) = -\cot \alpha, \quad (3.8)$$

then the solution (3.6) of equation (3.7) will satisfy the boundary condition (1.2)

$$\tilde{\varphi}(0, \lambda^2) \cos \alpha + \tilde{\varphi}'(0, \lambda^2) \sin \alpha = 0$$

for all $\lambda \in \mathbb{C}$. Since from (2.3) it follows that $G(0, 0) = -F(0, 0)$ and from (2.4) that $F(0, 0) = -\sum_{n=0}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{1}{a_n^0} \right)$, we have that condition (3.8) can be represented as

$$\sum_{n=0}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{1}{a_n^0} \right) = \cot \alpha,$$

which is our condition (1.8) on the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$.

It is also proved (see [1, 13]) that the expression

$$\frac{\tilde{\varphi}'_n(\pi)}{\tilde{\varphi}_n(\pi)} = \frac{\tilde{\varphi}'(\pi, \lambda_n^2)}{\tilde{\varphi}(\pi, \lambda_n^2)}$$

is a constant (i.e. does not depend on n), which we will denote by $-\cot \tilde{\beta}$. So the functions $\tilde{\varphi}(x, \lambda_n^2)$, $n = 0, 1, 2, \dots$, are the eigenfunctions of a problem $(q, \tilde{\alpha}, \tilde{\beta})$, where $q(x) = 2 \frac{d}{dx} G(x, x)$, $\tilde{\alpha}$ is in advance given α and we want $\tilde{\beta}$ to be equals β . We know from the Theorem 1.2, that for problem $(q, \alpha, \tilde{\beta})$ it holds

$$\frac{1}{\tilde{b}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi} \right) = -\cot \tilde{\beta}.$$

Thus, if we obtain condition (1.11), then we guarantee that $\tilde{\beta} = \beta$. But (1.11) deals with the norming constants \tilde{b}_n , which are not independent. We have shown that we can represent \tilde{b}_n by \tilde{a}_n and $\{\mu_k\}_{k=0}^{\infty}$ (see the relations (2.12) and (2.13)). Therefore, instead of (1.11), we obtain the condition in the form (1.9). Theorem 1.1 is proved.

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